Riemann Surfaces, Examples 1

Michaelmas 2020

Please send any comments or corrections to me at `hjrw2@cam.ac.uk`.

1. Let $U = \mathbb{C} \setminus ([-1,0] \cup [1,\infty))$ and let γ be a closed curve in U. Using standard properties of winding numbers, show that (i) $n(\gamma,1) = 0$, and (ii) $n(\gamma,0) = n(\gamma,-1)$.

2. Let $P(w_0, w_1, \ldots, w_s; z)$ be a polynomial in the $s+1$ complex variables $w_0, w_1, \ldots, w_s$, where the coefficients of P are holomorphic on $\mathbb{C}$. Thus
$$P(f(z), f^{(1)}(z), \ldots, f^{(s)}(z); z) = 0$$
is a differential equation, which we abbreviate to $P(f) = 0$. If (f, D) is a function element with $P(f) = 0$ in D and if $(g, D') \approx (f, D)$ is an analytic continuation, then show that $P(g) = 0$ in D'. Give an example of a differential equation and function elements as above, where $D' = D$ but $g \neq f$ on D.

3. Let $\pi : \tilde{X} \to X$ be a covering map of topological spaces (recalling here that the spaces are assumed connected and Hausdorff), and $f : \tilde{X} \to \tilde{X}$ a continuous map such that $\pi \circ f = \pi$. Show that f has no fixed points unless it is the identity.

4. Show that the power series $f(z) = \sum_{n>1} \frac{1}{n(n-1)} z^n$ defines an analytic function $(1-z)\log(1-z) + z$ on the unit disc D. Deduce that the function element (f, D) defines a complete analytic function on $\mathbb{C} \setminus \{1\}$, but does not extend to an analytic function on $\mathbb{C} \setminus \{1\}$.

5. Show that the power series $f(z) = \sum z^{2^n}/2^n$ has the unit circle as a natural boundary.

6. Let T be the complex torus $\mathbb{C}/\langle 1, \tau\rangle$, and let $Q_1 \subset \mathbb{C}$ be the open parallelogram with vertices $0, 1, \tau, 1+\tau$, and Q_2 the translation of Q_1 by $(1+\tau)/2$. Let U_1, U_2 denote the open subsets of T given by projection of Q_1, Q_2 respectively, and let $\phi_1 : U_1 \to Q_1$, $\phi_2 : U_2 \to Q_2$ be the charts obtained by taking the inverse maps. Describe explicitly the transition function
$$\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)\,.$$

7. By considering the singularity at ∞ or otherwise, show that any injective analytic map $f : \mathbb{C} \to \mathbb{C}$ has the form $f(z) = az + b$, for some $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$. Find the injective analytic maps $\mathbb{C}_\infty \to \mathbb{C}_\infty$.

8. Let $\Lambda = \langle \tau_1, \tau_2 \rangle$ be a lattice in $\mathbb{C}$ and let $T = \mathbb{C}/\Lambda$ be the corresponding complex torus. Let Λ' denote the lattice $\langle 1, \tau_2/\tau_1 \rangle$ and $T' = \mathbb{C}/\Lambda'$. Show that the Riemann surfaces T and T' are analytically isomorphic (i.e. conformally equivalent).

9. Define an equivalence relation $\sim$ on $\mathbb{C}^*$ by $z \sim w$ iff $z = 2^s w$ for some $s \in \mathbb{Z}$. Show that the quotient space $R = \mathbb{C}^*/\sim$ has the natural structure of a compact Riemann surface, and that R is analytically isomorphic to a complex torus.

10. (The identity principle for Riemann surfaces) Let R, S be Riemann surfaces, and $f, g : R \to S$ be analytic maps between them. Set $E = \{z \in R : f(z) = g(z)\}$; show that either $E = R$ or E contains only isolated points.

11. Let $D \subset \mathbb{C}$ be an open disc and u a harmonic function on D. Define a complex valued function g on D by $g = u_x - i u_y$; show that g is analytic. If z_0 denotes the centre of the disc, define a function f on D by

$$f(z) = u(z_0) + \int_{z_0}^{z} g\,,$$

the integral being taken over the straight line segment. Show that f is analytic with $f' = g$, and that $u = \mathrm{Re} f$.

12. Suppose u, v are harmonic functions on a Riemann surface R and $E = \{z \in R : u(z) = v(z)\}$. Show that either $E = R$, or E has empty interior. Give an example to show that E does not in general consist of isolated points.

13. Let $\{a_1, a_2, a_3, a_4\}$ and $\{b_1, b_2, b_3, b_4\}$ both be sets of four distinct points in $\mathbb{C}_\infty$. Show that any analytic isomorphism

$$f : \mathbb{C}_\infty \setminus \{a_1, a_2, a_3, a_4\} \to \mathbb{C}_\infty \setminus \{b_1, b_2, b_3, b_4\}$$

extends to an analytic isomorphism $\mathbb{C}_\infty \to \mathbb{C}_\infty$. Using your answer to Question 7, find a necessary and sufficient condition for $\mathbb{C} \setminus \{0, 1, a\}$ to be conformally equivalent to $\mathbb{C} \setminus \{0, 1, b\}$, where a, b are complex numbers distinct from 0 and 1.

14. Let $f(z)$ be the complex polynomial $z^3 - z$; consider the subspace R of $\mathbb{C}^2 = \mathbb{C} \times \mathbb{C}$ given by the equation $w^2 = f(z)$, where (z, w) denote the coordinates on $\mathbb{C}^2$, and let $\pi : R \to \mathbb{C}$ be the restriction of the projection map onto the first factor. Show that R has the structure of a Riemann surface, on which π is an analytic map. If g denotes the projection onto the second factor, show that g is also an analytic map.

    By deleting three appropriate points from R, show that π yields a covering map from the resulting Riemann surface $R_0 \subset R$ to $\mathbb{C} \setminus \{-1, 0, 1\}$, and that R_0 is analytically isomorphic to the Riemann surface (constructed by gluing) associated with the complete analytic function $(z^3 - z)^{1/2}$ over $\mathbb{C} \setminus \{-1, 0, 1\}$.

15. Let $f(z) = \sum a_n z^n$ be a power series of radius of convergence 1, and for w in the open unit disc, set $\rho(w)$ to be the radius of convergence for the power series expansion about w (so that $\rho(0) = 1$). Show that a point $\zeta \in C(0,1)$ on the unit circle is regular if and only if $\rho(\zeta/2) > \frac{1}{2}$. Suppose furthermore that all the a_n are non-negative real numbers. If $\zeta \in C(0,1)$, show that $|f^{(r)}(\zeta/2)| \leq f^{(r)}(1/2)$ for all r, and hence that $\rho(\zeta/2) \geq \rho(1/2)$. Deduce that 1 is a singular point.